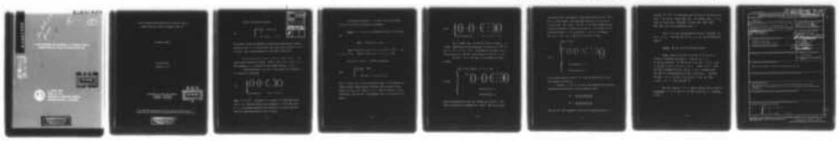
AD-A051 836 STANFORD UNIV CALIF DEPT OF OPERATIONS RESEARCH F/G 12/1
A FINITE PROCEDURE FOR DETERMINING IF A QUADRATIC FORM IS BOUND--ETC(U)
AUG 77 B C EAVES DAAG29-74-C-0032

UNCLASSIFIED ARO-12396.9-M NL

| OF |

AD
A051 836



END
DATE
FILMED
4-78
DDC

ARO 12396.9-M

See back page for 1473

5

A FINITE PROCEDURE FOR DETERMINING IF A QUADRATIC FORM IS BOUNDED BELOW ON A CLOSED POLYHEDRAL CONVEX SET

B. CURTIS EAVES
AUGUST 16, 1977
DEPARTMENT OF OPERATIONS RESEARCH
STANFORD UNIVERSITY, CALIFORNIA

A FINITE PROCEDURE FOR DETERMINING IF A QUADRATIC FORM IS BOUNDED BELOW ON A CLOSED POLYHEDRAL CONVEX SET

TECHNICAL REPORT

B. CURTIS EAVES

AUGUST 16, 1977

DEPARTMENT OF OPERATIONS RESEARCH
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

This research was supported in part by Army Research Office - Durham Contract DAAG-29-74-C-0032 and NSF Grant MPS-72-04832-A03.

Consider the quadratic program

$$(1)\qquad \begin{cases} V \triangleq \inf\limits_{x}: \; x \cdot Qx + x \cdot q \\ \\ \text{s/t:} \; Ax \leq a \qquad x \geq 0 \end{cases}$$

We describe a finite but inefficient procedure for determining the optimal objective value, V, of the program, and in particular, whether or not V is finite. This task was suggested to the author by David Gale.

Let Q be $n \times n$, q $n \times 1$, A $m \times n$, and a $m \times 1$. We assume that the program is feasible. Define $\mathcal{Q}(x)$ to be $x \cdot Qx + x \cdot q$. The expression $x \cdot u$ indicates the inner product between x and u.

A Kuhn-Tucker point (u,v,x,y) of the program (1) is defined to be a solution to the system

$$(2)\qquad \begin{cases} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} q \\ a \end{pmatrix} + \begin{pmatrix} Q' & A^T \\ -A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ \\ (u,v,x,y) \geq 0 \qquad u \cdot x = v \cdot y = 0 \end{cases}$$

where $Q' = Q + Q^T$. Of course, if (u,v,x,y) is a Kuhn-Tucker point, then x is a feasible solution to the program (1). On the other hand, if x is an optimal solution to the program (1), it can be shown that there is a Kuhn-Tucker point of form (u,v,x,y).

To determine the value of V we shall need the following result from the folklore of quadratic programming.

<u>Lemma 1</u>: If (u,v,x,y) is a Kuhn-Tucker point of the program then

$$\mathcal{Q}(x) = (1/2)\,(x \cdot q - y \cdot a)$$

<u>Proof</u>: Using (2) we have $0 = x \cdot q + x \cdot Q'x + x \cdot A^T y$, and $0 = y \cdot a - y \cdot Ax$. Hence $2x \cdot Qx + 2x \cdot q = x \cdot q - y \cdot a$ $\boxtimes$

Now for $k = 0,1,2,\ldots$ consider the programs

$$(3,k) \qquad \begin{cases} V_k \triangleq \min\limits_{x}: \quad \mathcal{Q}(x) \\ \\ \text{s/t:} \quad Ax \le a \quad x \ge 0 \quad ex \le k \end{cases}$$

where $e = (1,1,\ldots 1)$. For all sufficiently large k the program has a compact nonempty feasible region, and hence, has an optimal solution. Clearly $V_k \ge V_{k+1}$ and $\lim V_k = V$ as k tends to infinity. A Kuhn-Tucker point (u,v,w,x,y,z) of the program $(3,k)$ is a solution to the system

$$(4,k)\qquad \begin{cases} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} q \\ a \\ k \end{pmatrix} + \begin{pmatrix} Q' & A^T & e^T \\ -A & 0 & 0 \\ -e & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ (u,v,w,x,y,z) \geq 0 \qquad u \cdot x = v \cdot y = w \cdot z = 0 \end{cases}$$

Let $\ell = m+n+1$ and J be the set $\{1,\ell+1\} \times \{2,\ell+2\} \times \ldots \times \{\ell,\ell+\ell\}$. Observe that for any nonnegative (u,v,w,x,y,z) in $R^{2\ell}$ we have $u \cdot x = v \cdot y = w \cdot z = 0$, if and only if, for some α in J $(u,v,w,x,y,z)_\alpha = 0$, that is, $(u,v,w,x,y,z)_i = 0$ for all i in α .

For each α in J and large k we consider the linear program

$$(5,\alpha,k)\qquad \begin{cases} V_k^\alpha \triangleq \text{min: } (1/2)(x \cdot q - y \cdot a - kz) \\ \text{s/t: } \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} q \\ a \\ k \end{pmatrix} + \begin{pmatrix} Q' & A^T & e^T \\ -A & 0 & 0 \\ -e & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ (u,v,w,x,y,z) \geq 0 \\ (u,v,w,x,y,z)_\alpha = 0 \end{cases}$$

where the minimization is over the variables (u,v,w,x,y,z) . Note that if (u,v,w,x,y,z) is feasible for $(5,\alpha,k)$, then (u,v,w,x,y,z)

solves (4,k) and is, consequently, a Kuhn-Tucker point of (3,k). Therefore, in view of Lemma 1, for any optimal solutions (u,v,w,x,y,z) to (5,α,k) we have $V_k^\alpha = \mathcal{Q}(x)$. For each α the linear program (5,α,k) is either feasible for all sufficiently large k or infeasible for all sufficiently large k ; let us partition $J = J_F \cup J_I$ accordingly. Note that α is in J_F if and only if the linear program

$$
(6,\alpha) \quad \begin{cases} \text{sup:} \quad w + ez \\ \text{s/t:} \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} q \\ a \end{pmatrix} + \begin{pmatrix} Q' & A^T & e^T \\ -A & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ \qquad (u,v,w,x,y,z) \geq 0 \\ \qquad (u,v,w,x,y,z)_\alpha = 0 \end{cases}
$$

has an optimal objective value of $+\infty$ where the maximization is over the variables (u,v,w,x,y,z) .

Assuming α is in J_F we can use the simplex method treating k parametrically to generate in a finite number of steps

$$S^\alpha = (u_1^\alpha, v_1^\alpha, w_1^\alpha, x_1^\alpha, y_1^\alpha, z_1^\alpha)$$

$$T^\alpha = (u_2^\alpha, v_2^\alpha, w_2^\alpha, x_2^\alpha, y_2^\alpha, z_2^\alpha)$$

such that $S^\alpha + kT^\alpha$ optimizes (5,α,k) for all sufficiently large k .

Therefore $S^\alpha + kT^\alpha$ is a Kuhn-Tucker point of (3,k) for all sufficiently large k and we have $\mathcal{Q}(x_1^\alpha + kx_2^\alpha) = V_k^\alpha$. Furthermore, given α in J_F there is a fixed triple $(C_1^\alpha, C_2^\alpha, C_3^\alpha)$ such that $V_k^\alpha = \mathcal{Q}(x_1^\alpha + kx_2^\alpha) = C_1^\alpha k^2 + C_2^\alpha k + C_3^\alpha$ for all sufficiently large k .

Select β so as to lexicographically minimize $(C_1^\alpha, C_2^\alpha, C_3^\alpha)$ over all α in J_F . Then $V_k^\beta \leq V_k^\alpha$ for all α in J_F and all sufficiently large k .

Lemma 2: $V_k^\beta = V_k$ for all sufficiently large k .

Proof: Choose $\bar{k}$ so that a) for all $k \geq \bar{k}$ $(5,\alpha,k)$ is feasible or infeasible according to α being in J_F or J_I , b) $(5,\alpha,k)$ optimized by $S^\alpha + kT^\alpha$ for all $k \geq \bar{k}$ and α in J_F , and c) $V_k^\beta \leq V_k^\alpha$ for all $k \geq \bar{k}$ and α in J_F . Assume $k \geq \bar{k}$. Since $x_1^\beta + kx_2^\beta$ is feasible to (3,k), $V_k^\beta \geq V_k$. Let x optimize (3,k), then there is a Kuhn-Tucker point of form (u,v,w,x,y,z) . Therefore, for some α in J_F we have $(u,v,w,x,y,z)_\alpha = 0$ and $V_k = \mathcal{Q}(x) = 1/2(x \cdot q - y \cdot a - kz) \geq V_k^\alpha \geq V_k^\beta$ $\boxtimes$

Hence V_k^β tends to V as k tends to infinity and the result is established; $V = -\infty$ if $C_1^\beta < 0$ or if $C_1^\beta = 0$ and $C_2^\beta < 0$, otherwise, $V = C_3^\beta$.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

(18) ARO (19) 12396.9-M

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
12396.9-M

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
(6) A Finite Procedure for Determining if a Quadratic Form is Bounded Below on a Closed Polyhedral Convex Set.

5. TYPE OF REPORT & PERIOD COVERED
(9) Technical Rep[redacted]t.

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
(10) B. Curtis/Eaves

8. CONTRACT OR GRANT NUMBER(s)
(15) DAAG29-74-C-0032, NSF-MPS-72-04832

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Stanford University
Stanford, California 94305

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
(12) 8p.

11. CONTROLLING OFFICE NAME AND ADDRESS

12. REPORT DATE
(11) 16 Aug[redacted]77

13. NUMBER OF PAGES
5

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

The findings in this report are not to be construed as an official Department of the Army position, unless so designated by other authorized documents.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Consider the quadratic program

$$(1)\quad \begin{cases} V \triangleq \inf_x: \ x \cdot Qx = x \cdot q \\ \text{s/t:} \ Ax \le a \qquad x \ge 0 \end{cases}$$

We describe a finite but inefficient procedure for determining the optimal objective value, V , of the program, and in particular, whether or not V is finite.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified 402 766

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)